



Weak Centers and Bifurcation of Critical Periods in Reversible Cubic Systems

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Abstract—In this paper, we investigate nonhomogeneous cubic differential systems with reversibility, which guarantees that the systems have a center at the origin. We apply the Ritt-Wu method to process algebraic equations and inequalities using Maple V.3 on a computer. We give an inductive algorithm for computing the period coefficient polynomials, we find the structure of solutions of systems of algebraic equations corresponding to isochronous centers and weak centers of every finite order, and we derive conditions on the parameters under which the origin is an isochronous center or a weak center of finite order. We show that with appropriate perturbations local bifurcation of critical periods will occur from weak centers of finite order and isochronous centers. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Recently, being important in the study of Hilbert's 16th problem, weak focus (see [1,2]) of planar polynomial differential systems is researched actively. One often computes the *Lyapunov value* with the aid of computer algebra systems and symbolic computation techniques to determine how many limit cycles bifurcate from a weak focus, cf. [3,4]. Meanwhile, another class of equilibria of center type has been of interest because the monotonicity of periods of closed orbit surrounding a center is a nondegeneracy condition of subharmonic bifurcation for periodically forced Hamiltonian systems [5]. The origin is referred to as a *nondegenerate center* of a planar differential system if the system does not have a double zero eigenvalue at the origin. Suppose a planar differential system $\dot{x} = V_1(x, y, \lambda)$, $\dot{y} = V_2(x, y, \lambda)$, $\lambda \in \mathbf{R}^n$, has a nondegenerate center at the origin and let $P(r, \lambda)$ denotes the minimum period of the closed orbit passing through $(r, 0)$, a point in a sufficiently small open interval $J = (-\alpha, \alpha)$ of x -axis. The origin is called a *weak center of finite order k* of the system for the parameter value λ_* if $F(r, \lambda_*) = P(r, \lambda_*) - P(0, \lambda_*)$

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satisfies $F(0, \lambda_*) = F'(0, \lambda_*) = \dots = F^{(2k+1)}(0, \lambda_*) = 0$ and $F^{(2k+2)}(0, \lambda_*) \neq 0$, where the indicated derivatives of F are taken with respect to the first argument of F . The origin is called an *isochronous center* if $F^{(k)}(0, \lambda_*) = 0$ for all $k \geq 0$. In particular, the origin is called a *rough center* if $F^{(2)}(0, \lambda_*) = 0$. A *local critical period* is a period corresponding to a critical point of the period function $P(r, \lambda)$ which bifurcates from a weak center.

In 1989, Chicone and Jacobs [6] discussed weak centers for quadratic systems and answered the question as to how many critical periods there can be bifurcating from weak centers. Later, Rousseau and Toni [7] gave corresponding results for the special cubic system

$$\begin{aligned}\dot{x} &= -y + \sum_{j+l=3} a_{jl} x^j y^l, \\ \dot{y} &= x + \sum_{j+l=3} b_{jl} x^j y^l,\end{aligned}$$

with homogeneous nonlinearities of third degree. So far as we are aware, there are no more studies of weak centers and bifurcation of critical periods for cubic polynomial systems. In [6, 7], it is explicitly more difficult to deal with weak centers than with weak foci because computations and deductions for weak foci are always involved in the discussion of weak centers.

For general nonhomogeneous cubic differential systems, the discussion becomes more complicated because of the large number of parameters and the interaction of quadratic and cubic terms. In this paper, we investigate reversible cubic differential systems with nonhomogeneous nonlinearities. A planar differential system is said to be *reversible* if it is symmetric with respect to a line. Up to translation and rotation of coordinates, any reversible cubic differential system can be written as

$$\begin{aligned}\dot{x} &= -y + a_1 x^2 + a_2 y^2 + a_3 x^2 y + a_4 y^3, \\ \dot{y} &= x + b_1 xy + b_2 x^3 + b_3 xy^2,\end{aligned}\tag{1}$$

with parameter $(a_1, a_2, a_3, a_4, b_1, b_2, b_3) \in \mathbf{R}^7$. This symmetry ensures that equation (1) has a center at the origin, cf. [2]. We apply the Ritt-Wu method (see [8, 9]) to process algebraic equations and inequalities using Maple V.3 on a computer. We give an inductive algorithm for computing the period coefficient polynomials, we find the structure of solutions of systems of algebraic equations corresponding to isochronous centers and weak centers of every finite order, and we derive conditions on the parameters under which the origin is an isochronous center or a weak center of finite order. We show that with appropriate perturbations local bifurcation of critical periods will occur from weak centers of finite order and isochronous centers.

2. GENERAL ALGORITHM

Taking polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ in (1), we have

$$\dot{r} = \dot{x} \cos \theta + \dot{y} \sin \theta = r^2 G_2(\theta) + r^3 G_3(\theta),\tag{2}$$

$$\dot{\theta} = \frac{(\dot{y} \cos \theta - \dot{x} \sin \theta)}{r} = 1 + r H_1(\theta) + r^2 H_2(\theta),\tag{3}$$

where

$$\begin{aligned}G_2(\theta) &= a_1 \cos^3 \theta + (a_2 + b_1) \sin^2 \theta \cos \theta, \\ G_3(\theta) &= (a_3 + b_2) \cos^3 \theta \sin \theta + (a_4 + b_3) \cos \theta \sin^3 \theta, \\ H_1(\theta) &= (b_1 - a_1) \cos^2 \theta \sin \theta - a_2 \sin^3 \theta, \\ H_2(\theta) &= (b_3 - a_3) \cos^2 \theta \sin^2 \theta + b_2 \cos^4 \theta - a_4 \sin^4 \theta.\end{aligned}$$

Thus,

$$\frac{dr}{d\theta} = \frac{r^2 G_2(\theta) + r^3 G_3(\theta)}{1 + r H_1(\theta) + r^2 H_2(\theta)}.\tag{4}$$

LEMMA 1. Equation (4) is analytic and

$$\frac{dr}{d\theta} = r^2 G_2 + \sum_{k=3}^{\infty} r^k (G_2 A_{k-2} + G_3 A_{k-3}) \quad (5)$$

in a sufficiently small neighborhood of $r = 0$, where

$$A_0 = 1, \quad A_1 = -H_1, \quad A_k = -H_2 A_{k-2} - H_1 A_{k-1}, \quad \forall k \geq 2. \quad (6)$$

PROOF. For all θ the functions $H_1(\theta)$ and $H_2(\theta)$ are uniformly bounded. Thus, for sufficiently small r ,

$$\frac{1}{1 + rH_1 + r^2H_2} = \sum_{k=0}^{\infty} r^k A_k \quad (7)$$

is analytic. By comparison of coefficients we obtain a recursive relation (6), which determines all A_k . The remainder of this proof is simple. ■

Let $\lambda = (a_1, a_2, a_3, a_4, b_1, b_2, b_3) \in \mathbf{R}^7$. Consider the solution of (4) with $r(0, \lambda) = r_0 > 0$ in the form

$$r(\theta, \lambda) = \sum_{k=1}^{\infty} u_k(\theta, \lambda) r_0^k. \quad (8)$$

The initial condition implies

$$u_1(0, \lambda) = 1, \quad u_k(0, \lambda) = 0, \quad \forall k > 1, \quad \lambda \in \mathbf{R}^n. \quad (9)$$

Replacing r in (5) with series (8) and comparing coefficients of r_0^k , $k = 1, 2, \dots$, we get successively the following differential equations:

$$\begin{aligned} u'_1 &= 0, \\ u'_2 &= G_2 u_1^2, \\ u'_3 &= (G_2 A_1 + G_3) u_1^3 + 2u_1 u_2 G_2, \\ &\vdots \end{aligned} \quad (10)$$

where u'_k denotes $\frac{d}{d\theta} u_k(\theta, \lambda)$. With (9), we obtain their solutions

$$\begin{aligned} u_1(\theta) &= 1, \\ u_2(\theta) &= \int_0^\theta G_2(\xi) d\xi, \\ u_3(\theta) &= \int_0^\theta (G_3 + G_2(2u_2 - H_1)) d\xi, \\ &\vdots \end{aligned} \quad (11)$$

one by one. That is what we often do for weak foci.

Finally, we compute the period $P(r_0, \lambda)$ of the closed orbit $C(r_0)$ passing through $(r_0, 0)$. From (3) and (7), we have

$$\begin{aligned} P(r_0, \lambda) &= \int_{C(r_0)} dt = \int_0^{2\pi} \frac{1}{1 + rH_1 + r^2H_2} d\theta \\ &= \int_0^{2\pi} \left(1 + \sum_{k=1}^{\infty} r^k A_k \right) d\theta \\ &= 2\pi + \int_0^{2\pi} \sum_{k=1}^{\infty} r^k A_k d\theta. \end{aligned} \quad (12)$$

Meanwhile, from (8), we obtain the following power series expansion:

$$\begin{aligned} \sum_{k=1}^{\infty} r^k A_k &= A_1 u_1 r_0 + (A_1 u_2 + A_2 u_1^2) r_0^2 + (A_1 u_3 + 2A_2 u_1 u_2 + A_3 u_1^3) r_0^3 \\ &\quad + (A_1 u_4 + A_2 (u_2^2 + 2u_1 u_3) + 3A_3 u_1^2 u_2 + A_4 u_1^4) r_0^4 + \cdots \end{aligned} \quad (13)$$

Therefore,

$$P(r_0, \lambda) = 2\pi + \sum_{k=1}^{\infty} p_k(\lambda) r_0^k, \quad (14)$$

where

$$\begin{aligned} p_1(\lambda) &= \int_0^{2\pi} A_1 u_1 d\theta = - \int_0^{2\pi} H_1(\theta) d\theta = 0, \\ p_2(\lambda) &= \int_0^{2\pi} (A_1 u_2 + A_2 u_1^2) d\theta, \\ p_3(\lambda) &= \int_0^{2\pi} (A_1 u_3 + 2A_2 u_1 u_2 + A_3 u_1^3) d\theta, \\ p_4(\lambda) &= \int_0^{2\pi} (A_1 u_4 + A_2 (u_2^2 + 2u_1 u_3) + 3A_3 u_1^2 u_2 + A_4 u_1^4) d\theta, \\ &\vdots \end{aligned} \quad (15)$$

and $A_k, u_k, k = 1, 2, \dots$ are determined by (6) and (11). This gives an inductive algorithm for computing the *period coefficient polynomials* $p_2, p_4, \dots, p_{2k}, \dots$. A *Maple* program for this algorithm is given in the Appendix. Obviously, $P(0, \lambda) = 2\pi$. Theoretically, from the so-called period coefficient lemma [6], for $k \geq 1$, $p_{2k+1} \in (p_2, p_4, \dots, p_{2k})$, the ideal generated by p_{2i} , $i = 1, 2, \dots, k$, over $\mathbf{R}[\lambda_1, \dots, \lambda_n]$. Clearly, the fact that $p_2(\lambda_*) = p_4(\lambda_*) = \dots = p_{2k}(\lambda_*) = 0$ implies $p_{2k+1}(\lambda_*) = 0$ and the first $k > 1$ such that $p_k(\lambda_*) \neq 0$ is even. From the definition of weak center in Section 1, we obtain the following.

LEMMA 2. *If for a certain $\lambda_* \in \mathbf{R}^7$, there exists an integer $k \geq 1$ such that*

$$p_2(\lambda_*) = p_3(\lambda_*) = \dots = p_{2k+1}(\lambda_*) = 0 \quad \text{and} \quad p_{2k+2}(\lambda_*) \neq 0, \quad (16)$$

then for system (1) with $\lambda = \lambda_$ the origin is a weak center of order k . Otherwise, the origin is an isochronous center.*

The *Maple* program produces the following expressions for the period coefficient polynomials:

$$\begin{aligned} p_2 &= \frac{\pi}{12} (-5a_1 b_1 + 4a_1^2 - 3b_3 + 3a_3 + b_1^2 - a_2 b_1 + 10a_1 a_2 - 9b_2 + 9a_4 + 10a_2^2), \\ p_4 &= \frac{\pi}{1152} (-468a_1 a_2 b_2 - 2a_2 b_1^3 + 126a_1 a_2 b_1^2 + b_1^4 + 18a_2 b_2 b_1 + 700a_2^3 b_1 \\ &\quad - 828a_1^2 a_2 b_1 - 712b_1 a_1^3 + 369a_1^2 b_1^2 - 58a_1 b_1^3 + 198a_4 a_1 b_1 - \dots), \\ p_6 &= \frac{\pi}{1244160} (-204930a_1 b_2 b_3 b_1 + 914706a_1 b_2 a_3 b_1 + 1054296a_1^2 b_2 b_3 \\ &\quad - 5238a_1 a_2 b_3 b_1^2 + 576828a_1^2 a_2 b_3 b_1 + 13014a_3 a_2 b_3 b_1 + \dots), \\ &\vdots \end{aligned}$$

This requires a large-scale computation. For example, for $k = 1, 2, \dots, 6$ the number of terms of p_{2k} is, respectively, 10, 49, 168, 462, 1092, and 2310.

To distinguish weak centers of order k by Lemma 2, we need to solve the algebraic system (16). For simplicity, we can compute every $p_j(\lambda)$ modulo the ideal generated by those previous ones $p_2(\lambda), \dots, p_{j-1}(\lambda)$. In order to do so, noting that $\text{degree}(p_2, b_3) = 1$, $\text{coeff}(p_2, b_3) = -\pi/4$, we let

$$\begin{aligned} q_2 &= \text{primpart}(p_2) \\ &= 4a_1^2 + 3a_3 + b_1^2 - 5a_1b_1 - 3b_3 - a_2b_1 - 9b_2 + 10a_1a_2 + 9a_4 + 10a_2^2, \\ q_4 &= \text{primpart}(\text{prem}(p_4, q_2, b_3)) \\ &= 130a_2^3a_1 + 51a_1^2b_2 + 3a_2^2b_2 - 78a_1^3a_2 + 39a_1^3b_1 + 3a_1^2a_3 + 12a_1^2a_4 \\ &\quad - 36a_1^2a_2^2 + 33a_2^2a_3 + 282a_2^2a_4 + 42a_2^3b_1 - 12a_1^2b_1^2 + 3a_2^2b_1^2 \\ &\quad + 18a_3a_4 - 3b_1^2b_2 + a_1b_1^3 + 3a_4b_1^2 + 18a_3b_2 + 81a_4^2 - 28a_1^4 \\ &\quad + 120a_2^4 + 27b_2^2 + 246a_1a_4a_2 + 48a_1^2b_1a_2 + 36a_1a_3a_2 \\ &\quad - 6a_1b_1^2a_2 - 3a_3b_1a_2 + 69a_4b_1a_2 + 54a_2b_2a_1 \\ &\quad + 51a_2^2b_1a_1 - 3a_3b_1a_1 + 39a_4b_1a_1 + 6a_1b_2b_1, \end{aligned}$$

where $\text{primpart}(f)$ and $\text{prem}(f, g, x)$ are *Maple* functions, standing for primitive part and pseudo-remainder, respectively, for polynomials. In particular, as in [8,9], $\text{prem}(p_4, q_2, b_3)$ reduces p_4 modulo q_2 by substitution of the variable b_3 , i.e.,

$$(J(q_2))^s p_4 = \text{pquo}(p_4, q_2, b_3)q_2 + \text{prem}(p_4, q_2, b_3), \quad (17)$$

for a certain polynomial $\text{pquo}(p_4, q_2, b_3)q_2$ and $s \in \mathbf{Z}_+$, where $J(q_2) = \text{lcoeff}(q_2, b_3)$, the leading coefficient of q_2 with respect to b_3 , and the variable b_3 is eliminated by substitution given by the relation $q_2 = 0$. By the *Maple* procedure “factor” the above expressions of q_2 and q_4 is proved to be irreducible over \mathbf{Q} , so for the general reversible cubic system (1), we hardly deduce a very simple condition on the parameters from Lemma 2 even if we only consider weak centers of order one.

THEOREM 1. *The origin is a weak center of order one of the reversible cubic system (1) with $\lambda = \lambda_*$ if and only if $q_2(\lambda_*) = 0$ and $q_4(\lambda_*) \neq 0$.*

Clearly, the choice of $\lambda_* = (1, 2, 4, 5, 3, 6, 55/3)$ satisfies the condition in Theorem 1, where $q_2(\lambda_*) = 0$ and $q_4(\lambda_*) = 21007 \neq 0$. Generally, let $q_{2k} = \text{primpart}(\text{prem}(p_{2k}, q_2, b_3))$, $k = 3, 4, \dots$. Since $\text{lcoeff}(q_2, b_3) \neq 0$, by Lemma 2, we know that for $\lambda_* \in \mathbf{R}^7$ the origin is a weak center of order k ($k \geq 1$) if and only if $q_2(\lambda_*) = \dots = q_{2k}(\lambda_*) = 0$ and $q_{2k+2}(\lambda_*) \neq 0$.

For example, the reversible cubic system

$$\begin{aligned} \dot{x} &= -y + a_3x^2y + a_4y^3, \\ \dot{y} &= x + b_2x^3 + b_3xy^2 \end{aligned} \quad (18)$$

is just a case considered in [7] when

$$a_3 = -(e_6 - 3e_4), \quad b_3 = (e_6 - 3e_4), \quad a_4 = -(e_4 - e_5), \quad b_2 = (e_4 + e_5), \quad (19)$$

where e_4 , e_5 , and e_6 are real parameters and supposed not to be zero simultaneously for system (18) to have nontrivial nonlinearities. Taking the substitution (19) and letting $a_1 = a_2 = b_1 = 0$ in Theorem 1, we obtain $q_2 = -6e_6$ and $q_4 = 108(e_4^2 + e_5^2) - 36e_5e_6$. Clearly, $e_6 = 0$ implies $q_2 = 0$ and $q_4 = 108(e_4^2 + e_5^2) \neq 0$. Thus, system (18) with (19) has a weak center of order at most one, i.e., the origin is either a rough center when $e_6 \neq 0$ or a weak center of order one when $e_6 = 0$. This statement is identical with Rousseau and Toni's in Theorem 3.3 of [7].

3. ISOCHRONE AND WEAK CENTERS

As shown previously, the discussion in the general case with seven parameters seems difficult because the period coefficient polynomials are complicated and irreducible. However, specializing our studies at system (1) with small restrictions on quadratic terms, for the cubic system

$$S_\lambda : \begin{cases} \dot{x} = -y - ax^2 + ay^2 + a_3x^2y + a_4y^3, \\ \dot{y} = x - 2axy + b_2x^3 + b_3xy^2, \end{cases} \quad (a \neq 0), \quad (20)$$

we will obtain some interesting results about isochronous centers and weak centers. Here let $\lambda = (a, a_3, a_4, b_2, b_3) \in \mathbf{R}^5$ denote the bifurcation parameters and

$$\begin{aligned} S_I &= \{\lambda \in \mathbf{R}^5 \mid a_3 = b_3 = -3a_4 = -3b_2\}, \\ S_{II} &= \{\lambda \in \mathbf{R}^5 \mid a_3 = b_3, a_4 = b_2 = 0\}, \\ S_{III} &= \mathbf{R}^5 \setminus S_I \setminus S_{II}. \end{aligned}$$

System $(S_\lambda)_{\lambda \in \mathbf{R}^5}$ is called to have a center of Type I (respectively, II, III) if it is nonlinear and $\lambda \in S_I$ (respectively, S_{II} , S_{III}).

THEOREM 2. *For system (20), a center of Type I or II is an isochronous center; a weak center of Type III has order at most 4.*

PROOF. For $\lambda \in S_I$, the corresponding system (20) is in the form

$$\begin{aligned} \dot{x} &= -y - ax^2 + ay^2 - 3bx^2y + by^3, \\ \dot{y} &= x - 2axy + bx^3 - 3bxy^2. \end{aligned} \quad (21)$$

It can also be written into

$$dt = \frac{dz}{iz - az^2 + ibz^3} = dz \left(\frac{-i}{z} + O(z) \right),$$

where $z = x + iy$, i.e., the system satisfies the Cauchy-Riemann conditions. By the residue theorem the period function $P(r_0, \lambda)$ is constant. Thus, the system with a center of Type I is an isochrone system.

For $\lambda \in S_{II}$, the corresponding system can be written into

$$\begin{aligned} \dot{x} &= -y - ax^2 + ay^2 + bx^2y, \\ \dot{y} &= x - 2axy + bxy^2. \end{aligned} \quad (22)$$

Similar to (4), by taking the polar coordinate we have

$$\frac{dr}{d\theta} = \frac{r^2 \cos \theta (-a + br \sin \theta)}{1 - ar \sin \theta}. \quad (23)$$

Let $\rho = ar \sin \theta$, $\beta = b/a^2$. Equation (23) can be simplified into the form $\sin \theta d\rho = (\rho^2(1 - \beta\rho)/(1 - \rho) + \rho) d(\sin \theta)$, which is followed immediately by $(\rho/\sqrt{\beta\rho^2 - 2\rho + 1}) = k \sin \theta$, where k is a constant. With substitution $\rho = \tau/(1 + \tau)$ we obtain $\tau = \pm \sin \theta \sqrt{k/(1 + (1 - \beta)k \sin^2 \theta)}$, which is an odd and 2π -periodic function of θ . Therefore, as for (12),

$$\begin{aligned} P(r_0, \lambda) &= \oint_{C(r)} dt = \int_0^{2\pi} \frac{1}{\dot{\theta}} d\theta = \int_0^{2\pi} \frac{1}{1 - ar \sin \theta} d\theta \\ &= \int_0^{2\pi} \frac{1}{1 - \rho} d\theta = \int_0^{2\pi} (1 + \tau) d\theta = 2\pi + \int_0^{2\pi} \tau d\theta = 2\pi. \end{aligned}$$

This proves that the system with a center of Type II is also an isochrone system.

Finally, for $\lambda \in S_{III}$, we need solve the algebraic system $p_2 = p_4 = \dots = p_{2k} = 0$, $k = 1, 2, \dots, 5$. Take the ordering $b_3 \prec a_3 \prec b_2 \prec a_4$ and let $q_2 = p_2$ and q_{2k} be given from p_{2k} reduced modulo the ideal generated by $\{q_2, \dots, q_{2(k-1)}\}$. Computation with the *Maple* program implies that

$$\begin{aligned} q_2 &= -b_3 + a_3 - 3b_2 + 3a_4, \\ q_4 &= 2(b_2 + a_4)a_3 + 9a_4^2 + 3b_2^2, \\ q_6 &= (15a_4 + 8a^2)b_2^3 + (150a_4^2 - 28a_4a^2)b_2^2 \\ &\quad + (15a_4^3 + 32a_4^2a^2)b_2 + 12a^2a_4^3, \\ q_8 &= -132890625a_4^7 - 9071831250a^2a_4^6 + 15239407500a^4a_4^5 \\ &\quad + 2404621800a^6a_4^4 + 724745440a^8a_4^3 - 17954688a^{10}a_4^2 \\ &\quad - 1544704a^{12}a_4 + 184320a^{14}, \\ q_{10} &= a, \end{aligned}$$

where a common factor of constant multiple of π in each formula is omitted for convenience and the variables b_3 , a_3 , b_2 , and a_4 are eliminated in order step by step. The assumption that $a \neq 0$ implies $q_{10} \neq 0$. By the relation on zeros of algebraic equations (cf. [8,9]),

$$\text{Zero}(p_2, \dots, p_{2k}) = \text{Zero}\left(\frac{q_2, \dots, q_{2k}}{J_2, \dots, J_{2k}}\right) \cup \left(\bigcup_{i=1}^k \text{Zero}(p_2, \dots, p_{2k}, J_{2i})\right), \quad (24)$$

where J_2, \dots, J_{2k} are leading coefficients of q_2, \dots, q_{2k} , $\text{Zero}(p_2, \dots, p_{2k})$ consists of all common zeros of p_2, \dots, p_{2k} and $\text{Zero}(\{q_2, \dots, q_{2k}\}/\{J_2, \dots, J_{2k}\})$ consists of those common zeros of p_2, \dots, p_{2k} at which all J_2, \dots, J_{2k} do not vanish. Obviously, solutions of the system that $q_2 = 0, \dots, q_{2k} = 0$, $J_2 \neq 0, \dots, J_{2k} \neq 0$ satisfy the system $p_2 = 0, \dots, p_{2k} = 0$. Furthermore, we have the following.

LEMMA 3. For $a \neq 0$,

$$\begin{aligned} \text{Zero}\left(\frac{\{p_2\}}{p_4}\right) &= \text{Zero}\left(\frac{\{q_2\}}{q_4}\right), \\ \text{Zero}\left(\frac{\{p_2, p_4\}}{p_6}\right) &= \text{Zero}\left(\frac{\{q_2, q_4\}}{q_6}\right), \\ \text{Zero}\left(\frac{\{p_2, p_4, p_6\}}{p_8}\right) &= \text{Zero}\left(\frac{\{q_2, q_4, q_6\}}{q_8}\right), \\ \text{Zero}\left(\frac{\{p_2, p_4, p_6, p_8\}}{p_{10}}\right) &= \text{Zero}\left(\frac{\{q_2, q_4, q_6, q_8\}}{q_{10}}\right), \\ \text{Zero}(\{p_2, p_4, p_6, p_8, p_{10}\}) &= \emptyset. \end{aligned}$$

PROOF OF LEMMA 3. In the the ordering $b_3 \prec a_3 \prec b_2 \prec a_4$, we have $J_2 = \text{lcoeff}(q_2, b_3) = -1$, $J_4 = \text{lcoeff}(q_4, a_3) = 2(b_2 + a_4)$, $J_6 = \text{lcoeff}(q_6, b_2) = 15a_4 + 8a^2$, $J_8 = \text{lcoeff}(q_8, a_4) = -132890625$. Clearly, $\text{Zero}(\{p_2\}/p_4) = \text{Zero}(\{q_2\}/q_4)$. Observe that $J_4 \neq 0$ if $q_2 = 0$ and $q_4 = 0$. In fact, the assumption that $q_2 = q_4 = J_4 = 0$ implies $a_4 = b_2 = 0$ and $a_3 = b_3$, i.e., $\lambda \in S_{II}$. This is a contradiction with the fact that $\lambda \in S_{III}$. Hence, $\text{Zero}(\{p_2, p_4\}) = \text{Zero}(\{q_2, q_4\})$ and thus, $\text{Zero}(\{p_2, p_4\}/p_6) = \text{Zero}(\{q_2, q_4\}/q_6)$. Furthermore, $J_6 \neq 0$ if $q_2 = q_4 = q_6 = 0$. In fact, with the assumption that $q_2 = q_4 = q_6 = J_6 = 0$, the polynomial q_6 can be reduced modulo the ideal generated by $\{q_2, q_4, J_6\}$ to the polynomial

$$m_6 = -194400a^4b_2^2 + 23040a^6b_2 - 6144a^8.$$

The fact that $q_2 = q_4 = q_6 = J_6 = 0$ implies $m_6 = 0$. However, the discriminant $\Delta(m_6) = -4246732800a^{12} < 0$, which implies an opposite result that $m_6 \neq 0$ for $\lambda \in \mathbf{R}^5$. Similarly, we

can prove that $\text{Zero}(\{p_2, p_4, p_6\}/p_8) = \text{Zero}(\{q_2, q_4, q_6\}/q_8)$. The remaining two relations are explicit. ■

Now we continue to prove Theorem 2. For $k = 1, 2, 3, 4$, let

$$\Lambda_k := \{\lambda \in S_{\text{III}} \mid q_{2j}(\lambda) = 0, j = 1, \dots, k, \text{ and } q_{2k+2}(\lambda) \neq 0\}, \quad (25)$$

$$\Lambda_0 := \{\lambda \in S_{\text{III}} \mid q_2(\lambda) \neq 0\}. \quad (26)$$

Note that $q_{10}(\lambda) \neq 0$ even though $q_2(\lambda) = q_4(\lambda) = q_6(\lambda) = q_8(\lambda) = 0$. Clearly, $S_{\text{III}} = \bigcup_{k=0}^4 \Lambda_k$. From Lemma 3, we see easily that the origin is a weak center of order k if $\lambda \in \Lambda_k$, $k = 1, 2, 3, 4$, or a rough center if $\lambda \in \Lambda_0$. The proof of Theorem 2 is completed. ■

4. BIFURCATION OF CRITICAL PERIODS

In Section 1, the concept of local critical period is introduced. Strictly speaking, we say that k local critical periods bifurcate from the weak center corresponding to the parameter λ_* if for every sufficiently small $\alpha > 0$ there exists a neighborhood W of λ_* such that $P(r, \lambda)$ for any $\lambda \in W$ has at most k critical points in $U = (0, \alpha)$ and if any neighborhood of λ_* contains a point $\tilde{\lambda}$ such that $P(r, \tilde{\lambda})$ has exactly k critical points in $U = (0, \alpha)$. Furthermore, for $\lambda_* \in \text{Zero}(\{p_2, p_4, \dots, p_{2k}\}/p_{2k+2})$, the period coefficients p_2, \dots, p_{2k} of P are said to be independent with respect to p_{2k+2} at λ_* when the following conditions are satisfied:

- (i) every neighborhood of λ_* contains a point λ such that $p_{2k}(\lambda) \cdot p_{2k+2}(\lambda) < 0$;
- (ii) for any $j = 1, \dots, (k-1)$, if $\lambda \in \text{Zero}(\{p_2, p_4, \dots, p_{2j}\}/p_{2j+2})$ then every neighborhood of λ contains a point $\sigma \in \text{Zero}(p_2, p_4, \dots, p_{2j-2})$ such that $p_{2j}(\sigma) \cdot p_{2j+2}(\lambda) < 0$.

LEMMA 4. FINITE ORDER BIFURCATION THEOREM. (See [6].) From weak centers of finite order k at parameter λ_* no more than k local critical periods bifurcate. Moreover, there are perturbations with exactly j critical periods for any $j \leq k$, if the coefficients p_2, p_4, \dots, p_{2k} of F are independent with respect to p_{2k+2} at λ_* .

In this section, we discuss the bifurcation behaviors of critical periods in system (20) with $\lambda \in \Lambda_k$ for each $k = 1, 2, 3, 4$. We first prove the following result.

THEOREM 2. For $\lambda \in \Lambda_k$, where $k = 1, 2, 3$, or 4 , and for each $j = 1, \dots, k$, there exists a perturbation of (20) which has exactly j critical periods.

PROOF. Briefly, we only give the proofs for Λ_2 and Λ_4 . The discussion for Λ_1 and Λ_3 can be proceeded similarly.

For $\lambda_* \in \Lambda_2$, i.e., $\lambda_* = (a, a_4, b_2, a_3, b_3) \in \text{Zero}(\{p_2, p_4\}/p_6) = \text{Zero}(\{q_2, q_4\}/q_6)$, as in Lemma 3. From the construction of q_2, q_4, q_6 we have, up to a constant, that

$$\begin{aligned} q_2 &= p_2, \\ q_4 &= p_4 + A_{21}(\lambda)p_2, \\ q_6 &= J_4^2 p_6 + A_{32}(\lambda)p_4 + A_{31}(\lambda)p_2, \end{aligned} \quad (27)$$

for some polynomials $A_{ji}(\lambda)$ and positive integer s . It is clear that $q_2(\lambda_*) = q_4(\lambda_*) = 0$, $q_6(\lambda_*) \neq 0$ and from the proof of Lemma 3 that $J_4(\lambda_*) \neq 0$. Let $\lambda = (a, a_4, b_2, a_3 + \varepsilon, b_3 + \varepsilon)$. Then $J_4(\lambda) = J_4(\lambda_*)$, $q_6(\lambda) = q_6(\lambda_*)$, and

$$\begin{aligned} p_2(\lambda) &= q_2(\lambda_*) = 0, \\ p_4(\lambda) &= q_4(\lambda_*) + 2J_4(\lambda_*)\varepsilon = 2J_4(\lambda_*)\varepsilon, \\ p_6(\lambda) &= \frac{q_6(\lambda)}{J_4(\lambda)^s} - \frac{A_{32}(\lambda)}{J_4(\lambda)^s}p_4(\lambda) = \frac{q_6(\lambda_*)}{J_4(\lambda_*)^s} - 2\frac{A_{32}(\lambda_*)}{J_4(\lambda_*)^{s-1}}\varepsilon. \end{aligned}$$

Hence,

$$p_4(\lambda)p_6(\lambda) = 2 \frac{q_6(\lambda_*)}{J_4(\lambda_*)^{s-1}} \varepsilon - o(\varepsilon), \quad (28)$$

which means that in any neighborhood of λ_* there exists a parameter λ such that $p_4(\lambda)p_6(\lambda) < 0$. This proves the first condition of the independence. Furthermore, to verify the second condition, we assume $\lambda = (a, a_4, b_2, a_3, b_3) \in \text{Zero}(p_2/p_4)$. Let $\sigma = (a, a_4, b_2, a_3, b_3 + \varepsilon)$. Then

$$p_2(\sigma)p_4(\lambda) = -p_4(\lambda)\varepsilon. \quad (29)$$

Thus, we can choose ε small enough such that σ is contained in an arbitrarily given neighborhood of λ and $p_4(\sigma)p_6(\lambda) < 0$. This proves that p_2, p_4 is independent with respect to p_6 at every point in Λ_2 . According to Lemma 4, there is a perturbation which has exactly j critical periods for $j = 1, 2$.

For $\lambda_* \in \Lambda_4$, i.e., $\lambda_* \in \text{Zero}(\{p_2, \dots, p_8\}/p_{10})$, at first, we need check that every neighborhood of λ_* contains a point λ such that $p_8(\lambda)p_{10}(\lambda) < 0$. In fact, let $\lambda = (\lambda_i) := (a, a_4, b_2, a_3, b_3)$ for simplicity and

$$r_i = \text{resultant} \left(\frac{\partial p_8}{\partial \lambda_i}, \{q_2, \dots, q_8\} \right), \quad i = 1, \dots, 5, \quad (30)$$

which is given by the procedure

$$r_i = \text{resultant} \left(\dots \text{resultant} \left(\text{resultant} \left(\frac{\partial p_8}{\partial \lambda_i}, q_2, b_3 \right), q_4, a_3 \right), \dots, q_8, a_4 \right)$$

in the ordering $b_3 \prec a_3 \prec b_2 \prec a_4$, where $\text{resultant}(f, g, x)$ is a Maple function and called the *resultant* of f and g with respect to x . It is a simple result in [10] that the polynomials $\frac{\partial p_8}{\partial \lambda_i}, q_2, \dots, q_8$ are mutually prime, i.e., they have no common zeros, if and only if $r_i \neq 0$. For $\lambda_* \in \Lambda_4 \subset S_{\text{III}}$, as in the proof of Lemma 3, we know $a \neq 0$, $J_4(\lambda_*) \neq 0$ and $J_6(\lambda_*) \neq 0$. Thus,

$$\begin{aligned} \text{Zero} \left(p_2, \dots, p_8, \frac{\partial p_8}{\partial \lambda_1}, \dots, \frac{\partial p_8}{\partial \lambda_5} \right) &\subset \text{Zero} \left(q_2, \dots, q_8, \frac{\partial p_8}{\partial \lambda_1}, \dots, \frac{\partial p_8}{\partial \lambda_5} \right) \\ &\subset \text{Zero}(q_2, \dots, q_8, r_1, \dots, r_5) = \emptyset, \end{aligned} \quad (31)$$

where the final relation is proved by computing $\text{resultant}(q_2, \dots, q_8, r_1, \dots, r_5) \neq 0$ with *Maple V.3*. Hence, for $\lambda_* \in \Lambda_4$ there exists i , $1 \leq i \leq 5$, such that $\frac{\partial p_8}{\partial \lambda_i}(\lambda_*) \neq 0$. Choose $\lambda = \lambda_* + (\varepsilon_1, \dots, \varepsilon_5)$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_5)$ is small enough. We have

$$p_8(\lambda) = p_8(\lambda_*) + \frac{\partial p_8}{\partial \lambda_1}(\lambda_*)\varepsilon_1 + \dots + \frac{\partial p_8}{\partial \lambda_5}(\lambda_*)\varepsilon_5 + o(\varepsilon)$$

and $p_{10}(\lambda) \neq 0$. Therefore, we can choose ε such that λ is contained in an arbitrarily given neighborhood of λ_* such that

$$p_8(\lambda)p_{10}(\lambda) = \left(\frac{\partial p_8}{\partial \lambda_1}(\lambda_*)\varepsilon_1 + \dots + \frac{\partial p_8}{\partial \lambda_5}(\lambda_*)\varepsilon_5 \right) p_{10}(\lambda) + o(\varepsilon) < 0. \quad (32)$$

Next, we need check that, for any $j = 1, 2, 3$ and any $\lambda \in \text{Zero}(\{p_2, \dots, p_{2j}\}/p_{2j+2})$, every neighborhood of λ contains a point $\sigma \in \text{Zero}(\{p_2, \dots, p_{2j-2}\}/p_{2j})$ such that $p_{2j-2}(\sigma)p_{2j}(\lambda) < 0$. For $j = 1, 2$, this can be done in the same way as for Λ_2 . For $j = 3$, it is easy to prove

$$\text{Zero} \left(q_2, q_4, q_6, \frac{\partial p_6}{\partial a}, \frac{\partial p_6}{\partial a_4}, \frac{\partial p_6}{\partial b_2} \right) \subset \text{Zero} \left(q_6, \frac{\partial p_6}{\partial a}, \frac{\partial p_6}{\partial a_4}, \frac{\partial p_6}{\partial b_2} \right) = \emptyset, \quad (33)$$

provided $\lambda \in S_{\text{III}}$, where the last relation is proved by computing $\text{resultant}(q_6, \frac{\partial p_6}{\partial a}, \frac{\partial p_6}{\partial a_4}, \frac{\partial p_6}{\partial b_2}) \neq 0$ with *Maple V.3*. Given $\lambda = (a, a_4, b_2, a_3, b_3) \in \text{Zero}(\{p_2, p_4, p_6\}/p_8) = \text{Zero}(\{q_2, q_4, q_6\}/q_8)$, we

see that at least one of $\frac{\partial q_6}{\partial a}(\lambda)$, $\frac{\partial q_6}{\partial a_4}(\lambda)$, and $\frac{\partial q_6}{\partial b_2}(\lambda)$ is not zero and $J_4(\lambda) \neq 0$ as in Lemma 3. Let $\sigma = (a + \varepsilon_1, a_4 + \varepsilon_2, b_2 + \varepsilon_3, a'_3, b'_3)$ with $\varepsilon_1, \varepsilon_2, \varepsilon_3$ sufficiently small such that $J_4(\sigma) \neq 0$ and

$$\begin{aligned} q_2(\sigma) &= q_2(a + \varepsilon_1, a_4 + \varepsilon_2, b_2 + \varepsilon_3, a'_3, b'_3) = 0, \\ q_4(\sigma) &= q_4(a + \varepsilon_1, a_4 + \varepsilon_2, b_2 + \varepsilon_3, a'_3) = 0. \end{aligned}$$

Then by (27), we obtain that $p_2(\sigma) = 0$, $p_4(\sigma) = 0$, and $p_6(\sigma) = q_6(\sigma)/J_4(\sigma)^s$. Hence,

$$\begin{aligned} p_6(\sigma)p_8(\lambda) &= \frac{q_6(\sigma)}{J_4(\sigma)^s}p_8(\lambda) = q_6(a + \varepsilon_1, a_4 + \varepsilon_2, b_2 + \varepsilon_3)J_4(\sigma)^{-s}p_8(\lambda) \\ &= \left(\frac{\partial q_6}{\partial a}(\lambda)\varepsilon_1 + \frac{\partial q_6}{\partial a_4}(\lambda)\varepsilon_2 + \frac{\partial q_6}{\partial b_2}(\lambda)\varepsilon_3 \right) J_4(\sigma)^{-s}p_8(\lambda) + o(\varepsilon), \end{aligned}$$

so we can choose ε sufficiently small such that σ is contained in an arbitrarily given neighborhood of λ and $p_6(\sigma)p_8(\lambda) < 0$. This has verified the second condition of independence and the proof for Λ_4 is completed. ■

As above, only perturbations of weak centers of Type III are discussed, where results on finite order bifurcations of critical periods for system (20) are given. In the remaining cases of S_I and S_{II} , by Theorem 2 the center of (20) is isochronous, so we have to deal with bifurcations from isochronous centers. Consider a perturbation (E_λ) , $\lambda = (a, a_4, b_2, a_3, b_3)$, of system (E_{λ_*}) , $\lambda_* \in S_I \cup S_{II}$. Obviously, it suffices to discuss $\lambda = \lambda_* + \delta \in S_{III}$ because the perturbed system still displays an isochronous center when $\lambda \in S_I \cup S_{II}$. Unlike homogeneous cases in [6, 7], it seems very difficult to check for each integer $n \geq 1$ that the period coefficient p_{2n} of the nonhomogeneous system (E_λ) is in the ideal $(p_2, p_4, \dots, p_{2k}, p_{2k+2})$ over $\mathbf{R}[a, a_4, b_2, a_3, b_3]_{\lambda_*}$, the ring of convergent power series at λ_* , where k is a certain positive integer. We cannot apply the isochrone bifurcation theorem [6] directly to discuss the bifurcation from isochronous centers. However, the ideas in the proof of the isochrone bifurcation theorem in [6] enable us to prove the following.

THEOREM 4. *Consider system (E_{λ_*}) with $\lambda_* \in S_I \cup S_{II}$ and λ is in the neighborhood of λ_* . At most four critical periods bifurcate from the isochronous center in the family S_{III} .*

PROOF. This is to discuss bifurcations from the nonlinear isochronous center of system (E_{λ_*}) since $\lambda_* \neq 0$. $P(r, \lambda)$ being the period function, for $\lambda \in S_{III}$ we are going to find isolated positive zeros of the derivative $P'(r, \lambda)$ in r . Clearly, $P'(r, \lambda)$ has at most four more isolated positive zeros than

$$G(r, \lambda) = \frac{(((P'(r, \lambda)/r)'/r)'/r)'/r}{r} = 3640p_{10}(\lambda) + O(r), \quad (34)$$

where

$$p_{10}(\lambda) = \frac{\sum_{i=1}^4 Q_{2i}p_{2i}(\lambda)}{J_2^{s_2}J_4^{s_4}J_6^{s_6}J_8^{s_8}} + \frac{q_{10}(\lambda)}{J_2^{s_2}J_4^{s_4}J_6^{s_6}J_8^{s_8}} = \frac{\sum_{i=1}^4 Q_{2i}p_{2i}(\lambda_* + \delta)}{J_2^{s_2}J_4^{s_4}J_6^{s_6}J_8^{s_8}} + \frac{a}{J_2^{s_2}J_4^{s_4}J_6^{s_6}J_8^{s_8}}$$

and $Q_{2i} \in \mathbf{R}[a, a_4, b_2, a_3, b_3]$, $i = 1, \dots, 4$, as $\lambda \in S_{III}$. Note that $a \neq 0$ and $p_{2i}(\lambda_*) = 0$ for $\lambda_* \in S_I \cup S_{II}$, $i = 1, \dots, 4$. Moreover, from the proof of Lemma 3, we see that the leading coefficients J_2, J_4, J_6, J_8 do not vanish for $\lambda \in S_{III}$. By continuity it follows that $p_{10}(\lambda) \neq 0$ for all λ in the intersection of S_{III} with a sufficiently small neighborhood of λ_* . Hence, for λ in such an intersection, we can choose small $\varepsilon > 0$ such that in the interval $(0, \varepsilon)$ of r the function $G(r, \lambda)$ has no isolated zeros. Thus, $P'(r, \lambda)$ has at most four isolated positive zeros in $(0, \varepsilon)$. ■

REMARK. In the case of linear isochrone, there exist perturbations with the maximum number of critical periods. In fact, it is just the system (E_{λ_*}) where $\lambda_* = 0$. As in Theorem 3, its perturbations in S_{III} have each at most four critical periods and there is a perturbation in the form of (20) where $\lambda \in \Lambda_4$ with exactly four critical periods.

APPENDIX

MAPLE PROGRAM FOR PERIOD COEFFICIENT p_n

```
##### The main procedure #####
pols:=proc(n)
  local A,u,H1,H2,i,j,p,e:
  H1:=(b1-a1)*cos(t)^2*sin(t)-a2*sin(t)^3:
  H2:=(b3-a3)*cos(t)^2*sin(t)^2+b2*cos(t)^4-a4*sin(t)^4:
  A.(-1):=0: A.0:=1:
  for i to n do
  A.i:=-H2*A.(i-2)-H1*A.(i-1)
  od:
  G2:=a1*cos(t)^3+(a2+b1)*sin(t)^2*cos(t):
  G3:=(a3+b2)*cos(t)^3*sin(t)+(a4+b3)*cos(t)*sin(t)^3:
  u1:=1:
  for i from 2 to n do
  u.i:=Int(du(i),t): ":=value(":
  e:=subs(t=0,"):
  u.i:=normal("-e):
  od:
  Int(collect(coe(n),sin(t)),t=0..2*Pi):":=value(":
  p.m:=":
end:
##### Subprocedure for computing the coefficients of #####
##### the power series (13) #####
coe:=proc(n)
  local u,f,r,cd,ccd:
  f:=0:
  for i to n do
  f:=f+u.i*r^i:
  od:
  ccd:=0:
  for j to n do
  cd[j]:=f^j/(n!):
  for i to n do
  cd[j]:=diff(cd[j],r):
  od:
  f:=f-u.(n-j+1)*r^(n-j+1):
  ccd:=ccd+expand(subs(r=0,cd[j])*A.j):
  od:
end:
##### Subprocedure for differential equations (10) #####
du:=proc(n)
  local u,f,r,cd,ccd:
  f:=0:
  for k to n-1 do
  f:=f+u.k*r^k:
  od:
  ccd:=0:
  for j from 2 to n do
  cd[j]:=f^j/(n!):
  for k to n do
  cd[j]:=diff(cd[j],r):
  od:
  f:=f-u.(n-j+1)*r^(n-j+1):
```

```

ccd:=ccd+expand(subs(r=0,cd[j]))*(G2*A.(j-2)+G3*A.(j-3)):
od:
end:

```

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